EUROPEAN ACADEMIC RESEARCH
Vol. X, Issue 5/ August 2022

Impact Factor: 3.4546 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

A variational iteration method for solving non linear ordinary differential equations using the natural transform

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Abstract

In this article we have used the Variational Iteration Method (VIM) by adopting Natural transform in order to solve a general non- linear differential equations along with its four different types.

Keywords: Natural Transform, Inverse Natural transform, Variational Iteration method, Integral Transforms, non-linear ordinary differential equations,

1. INTRODUCTION

Natural transform (NT) is one of the most important integral transforms to solve differential and integral equations like other transforms. NT has various applications in science and engineering. It was introduced in 2008 by Zafar Hayat khan and Waqar khan. They applied Natural transform to the solution of linear differential equation of flow fluid type. They presented the significance of Natural transform that it converges to Laplace transform which is one of the oldest integral transforms (Khan, 2008) . The new developed integral transform become popular among researchers due to its dual nature of convergence. In 2011 Fethi Bin Muhammad and Silambaran applied Natural transform on Maxwell's equations (Belgacem, 2011) and in 2012 they highlighted some additional important properties of Natural transform which make Natural more comfortable for solving all types of differential equations as well as integral equations (Belgacem, 2012) . In 2016, Kamal Shah, Muhammad Junaid and Nigar Ali formed connection of some important integral transforms with Natural transforms. They extracted Laplace, Sumudu, Mellin and Fourier transform from the Natural transform. Their work indicates that Natural transform is also applicable to all the areas which previously were handled by pre-existed integral transforms (Shah, 2016), which provides the significant relation between Natural and other well-known transforms.

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Natural transform also useful to solve fractional differential equations (Ahmed, 2015) , (Loonker, 2013) also use ull in solving non-linear differential equations (Khandelwal, 2018). In this paper we have discussed the solution of non-linear ordinary differential equation using variational iteration method by Natural transform which is given in Result and Discussion section.

1.1 Definition

The Natural transform for function defined $\varphi(t) \geq 0$ for $t < 0$ is piecewise continuous and exponential order defined in the set

$$A = \left\{\varphi(t) : \exists M,\ \tau_1, \tau_2 > 0,\ |\varphi(t)| < Me^{\frac{t}{\tau j}}, if\ t \in (-1)^j \times [0, \infty)\right\} \#(1.1)$$

is given by

$$N[\varphi(t)] = R(s,u) = \int_0^\infty e^{-st}\varphi(ut)dt\ ;\ s > 0, u > 0 \#(1.2)$$

provided the integral on right side exists.
Where, s and u are transform variables, M is finite constant number and τ_1 and τ_2 could be finite or infinite
The inverse of Natural transform is defined as

$$N^{-1}[R(s,u)] = \varphi(t)$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s,u)ds \qquad (1.3)$$

2 RESULTS AND DISCUSSIONS

In this section we are presenting the kernel of fractional Sumudu transform and its relationship with fractional Fourier transform, which can play a significant role in solving complicated problems arising in signal processing and other fields of applied mathematics and will also helpful to obtained the solutions of fractional differential equation.

2.1 Method of variational iteration for solving non-linear ordinary differential equations by Natural transform.

Here, we have investigated the solution a general form of a non-linear ordinary differential equations (ODE) and its four types by method of variational method using Natural transform.
Let us consider the general form of non-linear ordinary differential equation

$$\frac{d^n y}{d^n t} + Ly(t) + Ny(t) = g(t) \quad \#(2.1)$$

Where $\frac{d^n v}{d^n t}$ is higher order derivative, L is a linear operator, N is a non-linear term and g is a known continuous function. We here first present the Variational iteration scheme in constructing the correctional function.
The Variational iteration method admits the use of correctional function for equation (2.1)

$$y_{k+1} = y_k + \int_o^t \lambda(\zeta)\left(\frac{d^n y}{d^n t} + Ly(\zeta) + Ny(\zeta) - g(\zeta)\right)d\zeta \#(2.2)$$

where $k = 0,1,2, \ldots$

Where $\lambda(\zeta)$ is said to be general Langrage multiplier
Now take the Natural transforms of eqn (2)

$$\mathrm{N}(y_{k+1}) = \mathrm{N}(y_k) + \mathrm{N}\int_o^t \lambda(t,\zeta)\left(\frac{d^n v}{d^n t} + Ly(t) + Ny(t) - g(t)\right) d\,\zeta,$$

$$R_{k+1}(s,u) = R_k(s,u) + \lambda(s,u)\left(\left(\frac{s}{u}\right)^n R_k(s,u) - \frac{u}{s^n} R_k(0)^{n-1} - \ldots - \left(\frac{s}{u}\right)^{n-1}(0) + N\big(U(v) + V(v) - g(t)\big)\right) \quad (2.3)$$

The value of λ can be obtained by considering $U(R(v) + NV(v)$ as restricted terms

$$\lambda = -\left(\frac{u}{s}\right)^n$$

Now, taking inverse Natural transforms of both sides of equation(2.3)

$$R_{k+1}(s,u) = R_k(s,u) - N^{-1}\left[\left(\frac{u}{s}\right)^n \left(\frac{s}{u}\right)^n R_k(s,u) - \frac{u}{s^n} R_k(0)^{n-1} - \ldots - \left(\frac{s}{u}\right)^{n-1}(0) + N\big(U(v) + V(v) - g(t)\big)\right],$$

$$= -N^{-1}\left[-\frac{u}{s^n} R_k(0)^{n-1} - \ldots - \left(\frac{s}{u}\right)^{n-1}(0) + N\big(U(v) + V(v) - g(t)\big)\right] \#(2.4)$$

Where the $y_o(t)$ can be obtained by

$$y_o(t) = N^{-1}\left[\left(\frac{u}{s}\right)^n R(0)^{n-1} + \ldots + \left(\frac{s}{u}\right)^{n-1}(0)\right]$$

Equation (2.4) indicates the Natural transform variational iteration technique.

2.1.1 Type-I

Consider the first order non- linear differential equation

$$\frac{dy}{dt} + ay^2 + b = 0 \#(2.5)$$

with initial condition $y_0(0) = 0$

Applying Natural transform variational method on eqn (5), we get,

$$y_{k+1}(t) = y_k + \int_o^t \lambda(t,\zeta)\left(\frac{dy}{dt} + ay^2 + b\right) d\,\zeta \#(2.6)$$

Taking Natural transform of eq (2.6) of both sides

$$N(y_{k+1}) = N\left[y_k + \int_o^t \lambda(t,\zeta)\left(\frac{dy}{dt} + ay^2 + b\right) d\,\zeta\right] \#(2.7)$$

$$R_{k+1}(s,u) = R_k(s,u) + \lambda(s,u)\left[\frac{s}{u} R_k(s,u) - \frac{y(0)}{u} + aN\big({y^2}_k\big) + \frac{b}{s}\right] \#(2.8)$$

Here,

$$\lambda(s,u) = -\frac{u}{s}$$

Eqn (2.8) becomes,

$$R_{k+1}(s,u) = R_k(s,u) - \frac{u}{s}\left[\frac{s}{u} R_k(s,u) - \frac{y(0)}{u} + aN({y_k}^2) + \frac{b}{s}\right]$$

$$R_{k+1}(s,u) = R_k(s,u) - \left[R_k(s,u) + \frac{au}{s} N({y_k}^2) + \frac{bu}{s^2}\right] \#(2.9)$$

Taking Natural inverse transform of above equation, we get

$$y_{k+1}(t) = y(t) - y(t) - bt - aN^{-1}\left(\frac{u}{s} N{y_k}^2\right),$$

$$y_{k+1}(t) = -bt - aN^{-1}\left(\frac{u}{s} N(y_k)^2\right) \#(2.10)$$

2.1.2 Type-II

Consider the nonlinear ODE of the form the form

$$\frac{dy}{dt} - y^2 - 1 = 0\#(2.11)$$

with initial condition $y_0(0) = 0$

Apply NVIM on eqn (4.11), we get,

$$y_{k+1}(t) = y_k + \int_o^t \lambda(t, \zeta)\left(\frac{dy}{dt} - y^2 - 1\right) d\,\zeta\#(2.12)$$

Taking Natural transform of eqn (2.12) of both sides

$$N(y_{k+1}) = N\left[y_k + \int_o^t \lambda(t, \zeta)\left(\frac{dy}{dt} - y^2 - 1\right) d\,\zeta\right]$$

$$R_{k+1}(s,u) = R_k(s,u) = \lambda(s,u)\left[\frac{s}{u} R_n(s,u) - \frac{y(0)}{u} - N(y^2) - \frac{1}{s}\right]\#(2.13)$$

Here,

$$\lambda(s,u) = -\frac{u}{s}$$

Eqn (2.13) becomes,

$$R_{k+1}(s,u) = R_k(s,u) - \frac{u}{s}\left[\frac{s}{u} R_n(s,u) - \frac{y(0)}{u} - N({y_k}^2) - \frac{1}{s}\right]$$

$$R_{k+1}(s,u) = R_k(s,u) - \left[R_k(s,u) - \frac{u}{s}N({y_k}^2) - \frac{u}{s^2}\right]\#(2.14)$$

Taking Natural inverse transform of above equation

$$y_{k+1}(t) = y(t) - y(t) + t + N^{-1}\left(\frac{u}{s}N{y_k}^2\right)$$

$$= t + N^{-1}\left(\frac{u}{s}N({y_k}^2)^2\right)\#(2.15)$$

If $k = 0$ then we get,

$$y_1(t) = t + N^{-1}\left(\frac{u}{s}N(y_0)^2\right),$$

$$y_1(t) = t + N^{-1}\left(\frac{u}{s}N(0)^2\right),$$

$$y(t) = t\#(2.16)$$

For $k = 1$,

$$y_2(t) = t + N^{-1}\left(\frac{u}{s}N(y_1)^2\right)$$

$$= t + N^{-1}\left(\frac{u}{s}N(t^2)\right)$$

$$= t + + N^{-1}\left(\frac{u}{s}\left(\frac{2u^2}{s^3}\right)\right)$$

$$= t + N^{-1}\left(\frac{2u^3}{s^4}\right)$$

$$y_2(t) = t + 2\frac{t^3}{3!}$$

$$y(t) = t + \frac{t^3}{3}\#(2.17)$$

For $k = 2$,

$$y_3(t) = t + N^{-1}\left(\frac{u}{s}N(y_2)^2\right)$$

$$= t + N^{-1}\left(\frac{u}{s}N\left(t + \frac{t^3}{3}\right)\right)^2$$

$$= \ t + N^{-1}\left(\frac{u}{s}N\left(t^2+\frac{t^6}{9}+\frac{2}{3}t^4\right)\right)$$

$$= t + N^{-1}\left(\frac{u}{s}\left(2\frac{u^2}{s^3}+80\frac{u^7}{s^8}+16\frac{u^5}{s^6}\right)\right)$$

$$= t + N^{-1}\left(2\frac{u^3}{s^4}+80\frac{u^8}{s^9}+16\frac{u^6}{s^7}\right)$$

$$= t+\frac{t^3}{3}+\frac{2}{15}t^5+\frac{t^7}{63}\#(2.18)$$

Continuing this manner, we will get the exact solution

$$\sum_{k=0}^{\infty}(-1)^{k-1}\,2^{2k}(2^{2k}-1)B_k t^{2k-1} = \tan(t)\ \#(2.19)$$

2.1.3 Type-III

Let us consider the ODE of following

$$\frac{dy}{dt} = -1+ty+y^2, \qquad y(0)=0\#(2.20)$$

$$y_{k+1}(t) = \ y_k + \int_o^t \lambda(\,t,\zeta)\left(\frac{dy}{dt}+1-ty-y^2\right)d\,\zeta\#(2.21)$$

Taking Natural transform of equation (2.21)

$$N[y_{k+1}(t)] = N\left[y_k + \int_o^t \lambda(\,t,\zeta)\left(\frac{dy}{dt}+1-ty-y^2\right)d\,\zeta\right],$$

$$R_{k+1}\,(u,s) = R_k\,(u,s) + \ \lambda(s)\left[\frac{s}{u}R_n(\,u,s) - \frac{y(0)}{u} + \frac{1}{s} - N(ty_k) - N(y_k^2)\right]\#(2.22)$$

Here, $\lambda(s) = \ -\frac{u}{s}$

$$= \ R_k\,(u,s) - \frac{u}{s}\left[\frac{s}{u}R_n(\,u,s) + \frac{1}{s} - N(ty_k) - N(y_k^2)\right] \qquad (2.23)$$

Taking inverse Natural of equation (2.23), we get

$$y_{k+1}(t) = \ -t + N^{-1}[N\,(ty_k) + \ N\,(y_k^2)]\#(2.24)$$

For $k=0,$ we get

$$y_1 = \ -t + \ N^{-1}[N\,(ty_0) + \ N\,(y_0^2)]$$

Let $y(0)=0$, then we have

$$y_1 = \ -t + \ N^{-1}[N\,(0) + \ N\,(0)]$$

$$y_1 = \ -t\#(2.25)$$

For $k=1,$

$$y_2 = \ -t + \ N^{-1}[N\,(ty_1) + \ N\,(y_1^2)]$$

$$y_2 = \ -t + \ N^{-1}[N\,(-t^2) + \ N\,(t^2)]$$

$$y_2 = \ -t\#(2.26)$$

For $k=2,$

$$y_3 = \ -t + \ N^{-1}[N\,(ty_2) + \ N\,(y_2^2)]$$

$$y_3 = \ -t + \ N^{-1}[N\,(-t^2) + \ N\,(t^2)]$$

$$y_3 = \ -t\#(2.27)$$

So, the exact solution of equation (4.20)is $y = \ -t$

2.1.4 Type-IV

Consider the nonlinear ODE for Isothermal Gas Sphere

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+e^y=0\#(2.28)$$

By applying NVIM on eqn (4.28), we get

$$=-\frac{u}{s^2}+N(ty_k)+N(y_k^2)\#(2.29)$$

$$N[y_{k+1}(t)]=N\left[y_k+\int_o^t\lambda(t,\zeta)\left(\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+e^y\right)d\,\zeta\right]\#(2.30)$$

$$R_{k+1}(u,s)=R_k(u,s)+\lambda(s)\left[\frac{s^2}{u^2}R_k(u,s)-\frac{y(0)}{u}-\frac{sY'(0)}{u^2}+N\left(\frac{2y'_k}{x}+e^{y_k}\right)\right] \quad (2.31)$$

Here $\lambda(s)=-\frac{u^2}{s^2}$ eqn (2.30)becomes

$$=R_k(u,s)-\frac{u^2}{s^2}\left[\frac{s^2}{u^2}R_k(u,s)+N\left(\frac{2y'_k}{x}+e^{y_k}\right)\right]$$

$$=-\frac{u^2}{s^2}N\left(\frac{2y'_k}{x}+e^{y_k}\right)\#(2.32)$$

Taking Inverse Natural of eqn (2.32)

$$y_{k+1}(t)=-N^{-1}\frac{u^2}{s^2}N\left(\frac{2y'_k}{x}+e^{y_k}\right)\#(2.33)$$

For $k=0$, the eqn(2.33) becomes

$$y_1(t)=-N^{-1}\frac{u^2}{s^2}N\left(\frac{2y'_0}{t}+e^{y_0}\right)$$

Let $y_0(0)=0$

$$=-N^{-1}\frac{u^2}{s^2}N(1)$$

$$=-N^{-1}\left(\frac{u^2}{s^3}\right)$$

$$y_1=-\frac{t^2}{2}\#(2.34)$$

For $k=1$

$$y_2(t)=-N^{-1}\frac{u^2}{s^2}N\left(\frac{2y'_1}{t}+e^{y_1}\right)$$

$$y_2(t)=-N^{-1}\frac{u^2}{s^2}N\left(-1-\frac{1}{2}t^2+\frac{3}{4}t^4-\frac{15}{6}t^6+\cdots\right)$$

$$y_2(t)=-N^{-1}\frac{u^2}{s^2}\left(-\frac{1}{s}-\frac{u^2}{s^3}+3\frac{u^4}{s^5}-15\frac{u^6}{s^7}\right)$$

$$=-N^{-1}\left(-\frac{u^2}{s^3}-\frac{u^4}{s^5}+3\frac{u^6}{s^7}-15\frac{u^8}{s^9}\right)$$

$$=\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{3t^6}{6!}+\frac{15t^8}{8!}+\ldots\#(2.35)$$

CONCLUSION:

A new approach to solve non-linear ordinary differential equation is established by using variational iteration method by Natural transform which indicate that the Natural transform is equally useful as compared to other integral transforms especially like Fourier and Laplace transforms for solving differential equations.

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