EUROPEAN ACADEMIC RESEARCH
Vol. IX, Issue 10/ January 2022

Impact Factor: 3.4546 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Calculating of Normal Curvature in Cylindrical Coordinates

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Abstract

In the present paper we dealt with some aspects of curvature and normal curvature. The aim of this study lies in Calculating of Normal Curvature in Cylindrical Coordinates numerically and using Matlab. Matlab is an interactive working environment in which user can carry out quite complex computational tasks with few commands. We followed the applied mathematical method usung a new mathematical technique (Matlab) and we found that the calculating of normal curvature in cylindrical coordinates using a new mathematical technique is more accurate and speed than the numerical calculating.

Keywords: normal curvature, cylindrical coordinates

1. INTRODUCTION

Differential geometry is a branch of mathematics using calculus to study the geometric properties of curves and surfaces. It arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces. The theory developed in this study originates from mathematicians of the 18th and 19th centuries, mainly ; Euler (1707-1783) , Monge (1746-1818) and Gauss (1777 - 1855).Mathematical study of curves and surfaces has been developed to answer some of the nagging and unanswered questions that appeared in calculus, such as the reasons for relationships between complex shapes and curves, series and analytic functions. Study of curvatures is an important part of differential geometry.[9,pp52]We will explore the concept of curvature and normal curvature in this study.

Definition(2.1): The locus of a point whose Cartesian coordinates (x,y,z) are functions of a single parameter is called curve and the locus of a point whose Cartesian coordinates (x,y,z) are functions of two independent parameters u ,v (say) is defined as a surface.[9,pp52]

Definition (2.2): Let S denote the shape operator of a surface M.

The Gauss (or Gaussian) curvature of Mat $p \in M$ is defined to be $K(p) = det(S_p)$.

The mean curvature of Mat $p \in M$ is defined to be $H(p) = \frac{1}{2}\, trace\,(S_p)$.[7,pp107]

Definition (2.2.) :

If γ is a unit speed curve with parameter t, its curvature $k(t)$ at the point $\boldsymbol{\gamma}(t)$ is defined to be $||\gamma''(t)||$.[2,pp30]

Definition (2.3) :

Let $\boldsymbol{\gamma} : \mathrm{I} \rightarrow \boldsymbol{R^n}$ be a regular curve. Its curvaturefunction, $k : I \rightarrow [0,\infty)$, is defined as

$$k\,(t) = \frac{|a^{\perp}(t)|}{|v(t)|^2}.\text{[8,pp26]}$$

Theorem (2.4): Let $k : [0, c] \rightarrow \boldsymbol{R^n}$ be a C^1 function. Then there exists a curve whose curvature function in the arc-length parameter is k. Such a curve is unique up to the choice of the initialpoint and the initial tangent vector.

Proof : Finding a curve $\gamma : [0, c] \rightarrow \boldsymbol{R^n}$parametrized by arc length and with curvature $k$is equivalent to solving the system of ordinary differential equations

$$\begin{cases} x''\,(s) = -\,k(s)y\,'(s) \\ y''(s) = \quad k(s)x\,'(s) \end{cases}$$

where $y\,(\mathrm{s}) = (\mathrm{x(s)}, \mathrm{y(s)})$ is the coordinate representation. Indeed, if the above equalities hold,then

$$x\,'(s)x''\,(s) = -\,k(s)y\,'(s)x\,'(s) = y\,'(s)y''(s)$$

and therefore $\|\gamma'(s)\|^2$ is constant. This is a linear first order system in(x'(s) ,$y'(s)$), and henceexistence and uniqueness are guaranteed by the standard ODE theory when we impose the initial condition (x'(0) , $y'(0)) = v$, where $v$is a unit vector. Since the tangent field $\gamma'(s)$ is now uniquely determined it follows (again by the standard ODE theory) that the curve is uniquely determined up to the choice of an initial point.[13,pp5]

Definition (2.5) :The Lie bracket *[X, Y]* of two vector fields *X*and *Y*on a surface *M*is defined as the commutator

$$[X, Y] = XY - YX \ ,$$

meaning that if f is a function on M, then $[X, Y](f) = X(Y(f)) - Y(X(f))$.

Proposition (2.6) :The Lie bracket of two vectors X , $Y \in T(M)$ is another vector in $T(M)$.

Proof: If suffices to prove that the bracket is a linear derivation on the space of C^∞ functions.

Consider vectors $X, Y \in T(M)$ and smooth functions f, gin M. Then,

$$\begin{aligned}[X, Y](f+g) &= X(Y(f+g)) - Y(X(f+g)) \\ &= X(Y(f)) - Y(X(f)) + X(Y(g)) - Y(X(g)) \\ &= [X, Y](f) + [X, Y](g),\end{aligned}$$

and

$$\begin{aligned}[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X[fY(g) + gY(f)] - Y[fX(g) + gX(f)] \\ &= X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f)) \\ &\quad -Y(f)X(g) - f(Y(X(g)) - Y(g)X(f) - gY(X(f)) \\ &= f[X(Y(g)) - (Y(X(g))] + g[X(Y(f)) - Y(X(f))] \\ &= f[X, Y](g) + g[X, Y](f).\end{aligned}$$

[6,pp52]

Theorem (2.7): Let *h(s)*be an arbitrary continuous function on a line segment *[a, b]*. Then there is a unique (up to a rigid motion) curve *γ*for which *h(s)*is the curvature function and *s* is the arc length parameter.

Proof: Let the functions *x(s), y(s)* and *α(s)*satisfy the system of equations

$$\frac{dx}{ds} = \cos \propto (s), \ \ \frac{dy}{ds} = \sin \propto (s), \ \ \frac{d \propto}{ds} = h\,(s).$$

Solving this system we get **:**

$$\propto (s) = \propto_0 + \int_0^s h(s)\, ds\ , \quad x(s) = x_0 + \int_0^s \cos\ \propto (s) ds,$$

$$y(s) = \ y_0 \ = \ \int_0^s \sin\ \propto (s)\ ds\,.$$

The obtained curve $\gamma : x = x(s),\ y = y(s)$satisfies all the conditions of the theorem. Prove that s is the arc length parameter. we have

$$l = \int_a^s \sqrt{(x')^2 + (y')^2}\, ds = \int_a^s ds = s - a$$

Further

$$|k(s)| = |x''(s)\vec{\imath} + y''(s)\vec{\jmath}| = \sqrt{(x'')^2 + (y'')^2} = \sqrt{|\propto'|^2} = \left|\frac{d\propto}{ds}\right| = |h(s)|.$$

In view of the definition of the sign of curvature we obtain $k(s) = \frac{d\propto}{ds} = h(s)$.

Finally, the coordinates of the initial point on the curve γ *(s)* are actually (x_0, y_0) and the direction of the tangent vector $\vec{\tau}(0)$forms the angle α0 with the *OX* axis.

Hence if there exist two curves with equal curvatures then a rigid motion that matches their initial points and initial tangent vectors at this point also maps one curve to the other. [12,pp23]

3. Normal Curvature: Although the shape operator does the job of measuring the bending of a surface in different directions, frequently it is useful to have a real-valued function, called the normal curvature, which does the same thing. We shall define it in terms of the shape operator, though it is worth bearing in mind that normal curvature is explicitly a much older concept .

Definition (3.1) :A drectionℓ on a regular surface *M* is a 1-dimensional subspace of (that is, a line through the origin in) a tangent space to *M*.

Definition (3.2) : Let u_p be a tangent vector to a regular surface $M \subset R^3$ with

$||u_p|| = 1$. Then the normal curvature of *M* in the direction u_p is $k(u_p) = S(u_p) \cdot u_p$.

More generally, if v_p is any nonzero tangent vector to *M* at$p$, we put

$k(v_p) = \frac{S(v_p).v_p}{\|v_p\|^2}$

If ℓ is a direction in a tangent space M_p to a regular surface $M \subset R^3$, then k(v_p) is easily seen to be the same for all nonzero tangent vectors v_p in ℓ. Therefore, we call the common value of the normal curvature the normal curvature of the direction ℓ.

Lemma (3.4): (Meusnier) Let u_p be a unit tangent vector to *M* at$p$, and letbe a unit-speed curve in *M* with $\beta(0) = p$ and $\beta'(0) = u_p$. Then

$$k(u_p) = k[\beta](0)\, cos\theta$$

Where $k[\beta](0)$is the curvature of β at 0, and is the angle between the norma *N(0)* of β and the surface normal *U(p)*. Thus all curves lying on a surface *M* and having the same tangent line at a given point $p \in M$ have the same normal curvature at p.[1,pp389]

Definition (3.5): Let $p$be a point ofM $\subset \boldsymbol{R}^3$. The maximum and minimum values of the normal curvature $k(\boldsymbol{u})$ of Mat $p$are called the principal curvatures of *M* at$p$, and are denoted by k_1 and k_2. The directions in which these extreme values occur are called principal directionsof *M* at$p$. Unit vectorsin these directions are called principal vectorsof *M* at$p$.[4,pp212]

Theorem (3.6) :(Meusnier's theorem)

Let S$\subset$ R^3 be an orientable regular surface with unit normal field N and second fundamental form II. Let p $\in$S.

Let c :(−ε, ε) → S be a curve parametrised by arc-length with c(0) = p. Thenwe have for the normal curvature k_{nor} of c : k_{nor} = II ($c'(0)$,$c'(0)$).

In particular, all curves parametrised by arc-length in S through p with the same tangent vector have the same normal curvature.

This also justifies that we refer to k_{nor} as the normal curvature of *S* at the point p in direction$c'(0)$, since k_{nor} depends apart from on *S* and p only on $\acute{c}(0)$ but not on the particular choice of the curve *c*.

Proof :

As c lies on *S* we have⟨N(c(t)), $c'(t)$⟩= 0

for all $t \in(-\varepsilon, \varepsilon)$. Differentiating this equation gives :

$$\begin{aligned} 0 &= \frac{d}{dt}\langle N\left(c(t)\right), c'(t)\rangle|_{t=0} \\ &= \langle \frac{d}{dt} N\left(c(t)\right)|_{t=0} \, . c'(0)\rangle + \langle N(p)\, , c''(0)\rangle \\ &= \langle d_p N\left(c'(0)\right). c'(0)\rangle + k_{nor} \\ &= \langle -W_p\left(c'(0)\right). c'(0)\rangle + k_{nor} \\ &= \langle -II\left(c'(0)\right). c'(0)\rangle + k_{nor} \end{aligned}$$

[5,pp111]

Some Important Information about Normal Curvature:

i. At any point *P* of a sufficiently smooth surface *S*, there exists a paraboloid tangent at its vertex to the tangent plane of *S* at *P* such that the normal curvature of the paraboloid in a given direction at *P* is equal to the normal curvature of *S* at P in that direction.

ii. The normal curvature k_n is an extrinsic property since it depends on the second fundamental form coefficients.

ii.The normal curvatures of surface curves at a given point *P* in the directions of the *u* and *v* coordinate curves are given respectively by $\frac{b_{11}}{a_{11}}$and $\frac{b_{22}}{a_{22}}$ where these are evaluated at *P*.

iv. The necessary and sufficient condition for a given point *P*on a sufficiently smooth surface *S*to be umbilical point is that the coefficients of the first and second fundamental forms of the surface are proportional that is:

$$\frac{e}{E} = \frac{f}{F} = \frac{g}{G} \; (k_n)$$

WhereE , F , G , e , f and g are the coefficients of the first and second fundamental forms at *P*, and $k_n$is the normal curvature of *S*at *P*in any direction.[11,pp96]

Theorem (3.7): Let *S* be a regular surface and$p \in S$. Let k_1, k_2 be the minimum and maximum normal curvatures at $p$and e_i their associated principal directions. Then let v be some unit vector in $\boldsymbol{R^2}$. Then for some $\theta \epsilon [0,2\pi)$,$v = e_1 \cos\theta + e_2 \sin\theta$

and the normal curvature in the direction *v*is given by :

$$k_n = k_1 cos^2 \theta + k_1 sin^2 \theta$$

Proof :Since the principal directions (e_1 , e_2) form an orthonormal basis of $\boldsymbol{R^2}$, any unit vector v can be expressed in the form $e_1 \cos\theta + e_2 \sin\theta$for some θ. This is the same way that any unit vector - identified as a point on the unit circle - has an angle $0 \leq \theta < 2\,\pi$associated with it. Further more , $d\,N_p(e_1) = -k_1 e_1$ and $d\,N_p(e_2) = -k_2 e_2$.

From our previous expression of normal curvature $k_n$in terms of $d\,N_p$:

$$\begin{aligned} k_n &= -\langle d\,N_p(v), v\rangle \\ &= -\langle d\,N_p(e_1 \cos\theta + e_2 \sin\theta) , e_1 \cos\theta + e_2 \sin\theta\rangle \\ &= -\langle - k_1(e_1 \cos\theta + k_2 e_2 \sin\theta) , e_1 \cos\theta + e_2 \sin\theta\rangle \\ &= k_1 cos^2 \theta \langle e_1, e_1\rangle + k_1 \cos\theta \sin\theta \langle e_1, e_2\rangle + k_2 \cos\theta \sin\theta \langle e_1, e_2\rangle \\ &\quad + k_2 \, sin^2 \theta \langle e_2, e_2\rangle \\ &= k_1 cos^2 \theta + k_2 \, sin^2 \theta \end{aligned}$$.[3,pp9]

4. METHODOLOGY AND DISCUSSION:

i. Matlab Functions:

Matlab has potpourris of function. Some of these are stander functions including trigonometric function etc., and others are user-defined function and third party function. All of these enable user to carry out complex computational tasks easily [10,pp7]

Some Elementary Functions:

Typically used common include sqrt ,exp , log and log 10 . Not that log function gives the natural logarithm. So

```
>>   x = 2 ;sqrt (x) , exp (x) , log (x) , log 10 (x)
ans=
        1.4142
ans =
         0.1351
ans =
         0.6931
ans =
         0.3010
```

Here all four function have been tested using the same command . As you can see the semicolon suppresses the ech while the comma separates various computations.[10,pp8]

Matlab ODE Solvers:

In addition to the many variations of the predictor-corrector and Runge-Kutta algorithms that have beeneveloped, there are more-advanced algorithms that use a variable step size. These "adaptive" algorithms use largerstep sizes when the solution is changing more slowly. Matlab provides several functions, called *solvers*, that implement the Runge-Kutta and other methods with variable step size. Two of these are the ode45 and ode15sfunctions. The ode45 function uses a combination of fourth- and fifth-order Runge-Kutta methods. It is ageneral-purpose solver, whereas ode15s is suitable for more-difficult equations called "stiff" equations. Thesesolvers are more than sufficient to solve the problems in this text. It is recommended that you try ode45 first. Ifthe equation proves difficult to solve (as indicated by a lengthy solution time or by a warning or error message),then use ode15s.we limit our coverage to first-order equations. Solution of higher-order equations is covered . When used to solve the equation , the basic syntax is (using ode45 as the example)[t,y] = ode45(@ydot, tspan, y0)where @ydot is the handle of the function file whose inputs must be *t* and *y*, and whose output must be a columnvector representing *dy/dt*, that is, *f*(*t*, *y*). The number of rows in this column vector must equal the order of theequation. The syntax for ode15s is identical. The function file ydot may also be specified by a character string(i.e., its name placed in single quotes), but use of the function handle is now the preferred approach.

The vector tspan contains the starting and ending values of the independent variable *t*, and optionally anyintermediate values of *t* where the solution is desired. For example, if no intermediate values are specified, tspanis [t0 tfinal], where t0 and tfinalare the desired starting and ending values of the independent parameter$t$. As another example, using tspan = [0, 5, 10] tells Matlab to find the solution at *t* = 5 and at *t* = 10.

You can solve equation backward in time by specifying t0 to be greater than tfinal.

The parameter y0 is the initial value *y*(0). The function file must have its first two input arguments as t and yin that order, even for equations where *f*(*t*, *y*) is not a function of *t*. You need not use array operations in the function file because the ODE solvers call the filewith scalar values for the arguments.

First consider an equation whose solution is known in closed form, so that we can make sure we are using themethod correctly.[14,pp374]

Example(5.1):

Use (a certain numerical method) to find the normal curvature $K_n(\theta)$ for the cylinder $r(u, v) = \langle cos(u), sin(u), v\rangle$ at the point $r(0, 0) = (1,0, 0)$

Solution: The curves $\rho_\theta(t) = r\ (t\cos(\theta)\ , t\sin(\theta)\ )$are given by

$$\rho_\theta(t) = \langle \cos(t\cos(\theta)\ , \sin(t\ \cos(\theta)\ , t\sin(\theta)\rangle$$

Differentiation with respect to t leads to

$$\rho'_\theta(t) = \langle -\sin(t\cos(\theta)\cos(\theta)\ , \cos(t\ \cos(\theta)\cos(\theta)\ , \sin(\theta)\rangle$$

$$\rho''_\theta(t) = \langle -\cos(t\cos(\theta)\, cos^2(\theta)\ , -\sin(t\ \cos(\theta)\, cos^2(\theta)\ , 0\rangle$$

so that at t = 0 we have:

$$\rho'_\theta(0) = \langle -\sin(0)\quad \cos(\theta)\,,\quad \cos(0)\cos(\theta)\ , \sin(\theta)\rangle = \langle 0\,, \cos(\theta)\,, \sin(\theta)\rangle$$

$$\rho''_\theta(t) = \langle -\cos(0)\quad cos^2(\theta)\ , -\sin(0)\, cos^2(\theta)\ , 0\rangle = \langle -cos^2(\theta)\ , 0\,, 0\rangle$$

The partial derivatives of $r(u,v)$ are $r_u = \langle -\sin(u)\ ,\cos(u)\ ,0\rangle, r_v = \langle 0,0,1\rangle$ which are both unit vectors. It follows that $r_u(0,0) = \langle 0,1,0\rangle =$ j and
$r_v = \langle 0,1,0\rangle =$ k. Thus:

$$n = r_u(0,0) \times r_v(0,0) = \mathrm{j} \times k = i$$

Since$\rho'_\theta(0)$ is a unit vector we have :

$$k_n(\theta) = \frac{\rho'_\theta(0)\ .n}{\left\|\rho'_\theta(0)\right\|^2} = \frac{\langle -cos^2(\theta)\ ,0\ ,0\rangle\ .i}{1} = -cos^2(\theta)$$

Because the normal curvature $k_n(\theta)$ is a real-valued continuous function over θ in $[0,2\,\pi]$ there is a largest $k_1$and a smallest $k_2$curvature at each point.

The numbers k_1 and k_2 are known as the principle curvatures of the surface$\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v})$.

In example (5.1), the largest possible curvature is $k_1 = -cos^2\left(\frac{\pi}{2}\right) = 0$in the vertical direction and the smallest possible curvature is $k_2 = -cos^2(0) = -1$ in the horizontal direction.

The average of the principal curvatures is called the Mean curvature of the surface and is denoted by *H*: It can be shown that if *C* is a sufficiently smooth closed curve then the surface with *C* as its boundary curve that has the smallest possible area must have a mean curvature of *H = 0*.

For example, consider two parallel circles in 3 dimensional spaceThe catenoid is the surface connecting the two circles that has the leastsurface area, and a catenoid also has a mean curvature of *H = 0*.

Surfaces with a mean curvature of *H = 0* are called minimal surfaces because they also have the least surface area for their given boundary. For example a soap film spanning a wire loop is a minimal surface. Moreover minimal surfaces have a large number of applications in architecture mathematics and engineering.

Example (5.2):

Use (a certain numerical method by Matlab) to find the normal curvature *k(θ)* for the cylinder *r(u, v) = ⟨cos(u), sin(u), v⟩*

Solution:

```
clc
clearall
symsuvrtr(u,v)pf(u,v)
```

```
r(u,v) = cos(u)
f(u,v)=sin(u)
% ??(t) = (cos(t cos(?)), sin(t cos(?)), t sin(?))
t = -1:1:5;
figure
[X,Y,Z] =cylinder
% [X,Y,Z] = cylinder(sin(-cos(t))^2);
surf(X,Y,Z)
axissquare
gridon
```

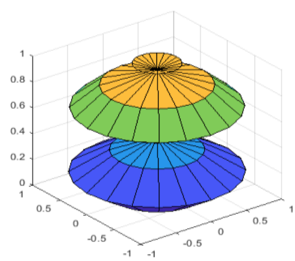

Fig. No.(1) : The Matlab Calculating

RESULTS:

After we showed the calculation of normal Curvature in cylindrical coordinates using a new mathematical technique we found the following some results :We showed that a new mathematical technique gives us a precise results of high speed compared with that of numerical, also we stated the ability capability of graphs or diagram drawing to any normal curvature via a new mathematical technique , we explained the possibility of the calculation of normalcurvature by a new mathematical technique with a very high rate and accuracy finally we can considered a new mathematical technique as atheory which is considered one of the most important mathematical techniqueto calculate the normal curvature and the other mathematical conceptions.

CONCLUSION:

Finally we can say that the method which we used in this paper help us in finding the most accurate results and drawing them in a more rapid , attractive and clear way . Therefore we hope that researchers will use this method (A New Mathematical Technique NMT) in their scientific papers.

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