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Implementation and Development of Multistep Algorithm Arising in Applied Sciences and Engineering

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Abstract:

As we cannot integrate every function so it is not possible to solve every differential equation. Even the differential equation of first order and first degree cannot be solved in every case. They can be solved, however, if they belong to standard forms. During the past half-century, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science and engineering, and numerical analysis of increasing sophistication has been needed to solve these more detailed mathematical models of the world. The formal academic area of numerical analysis varies from quite theoretical mathematical studies to computer science issues.

Key words: Multistep Algorithm, Applied Sciences, Engineering

1.1 Introduction:

Linear multistep methods are used for the numerical solution of ordinary differential equations. Conceptually, a numerical method starts from an initial point and then takes a short **step** forward in time to find the next solution point. The process continues with subsequent steps to map out the solution. Single-step methods (such as Euler's method) refer to

only one previous point and its derivative to determine the current value. Methods such as Runge-Kutta take some intermediate steps (for example, a half-step) to obtain a higher order method, but then discard all previous information before taking a second step. Multistep methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, multistep methods refer to several previous points and derivative values. In the case of *linear* multistep methods, a linear combination of the previous points and derivative values is used. In this research paper we are solving ordinary differential equation arising in electrical circuit and Rate of cooling by multiple step method and comparing the result by analytical method. The organization of this paper is as follow in section 1.2 we are describing Milne's method, in section 1.2 we are describing Adam's Bashford Method finally we are implementing the multistep method in electrical circuit and Rate of cooling and comparing the result obtained by multistep method with analytical solution

1.2 Milne's method

The methods of Euler, Heun, Taylor and Runge-Kutta are called single-step methods because they use only the information from one previous point to compute the successive point, that is, only the initial point (t_0, y_0) is used to compute (t_1, y_1) and in general y_k is needed to compute y_{k+1}. After several points have been found it is feasible to use several prior points in the calculation. The Milne-Simpson method uses $y_{k-3}, y_{k-2}, y_{k-1}$ *and* y_k in the calculation of y_{k+1}. This method is not self-starting; four initial points $(t_0, y_0), (t_1, y_1), (t_2, y_2), and\ (t_3, y_3)$ must be given in advance in order to generate the points $\{(t_k, y_k)\}k^m = 4$.

A desirable feature of a multistep method is that the local truncation error (L. T. E.) can be determined and a correction term can be included, which improves the accuracy of the answer at each step. Also, it is possible to determine if the step size is small enough to obtain an accurate value for y_{k+1}, yet large enough so that unnecessary and time-consuming calculations are eliminated. If the code for the subroutine is fine-tuned, then the combination of a predictor and corrector requires only two function evaluations of f(t , y) per step. It requires past four points of the solution to predict the fifth value. The actual curve is approximated by a fourth degree polynomial.

The Newton's formula can be written as

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0 + \ldots (4.1)$$

Where $u = \frac{x - x_0}{h}$ *or* $x = x_0 + hu.$

For = $y = y'$, this gives

$$y' = y_0^{'} + u\Delta y_0^{'} + \frac{u(u-1)}{2}\Delta^2 y_0^{'} + \frac{u(u-1)(u-2)}{6}\Delta^2 y_0^{'} + \ldots$$

Integrating (1) over the interval x_0 to x_0 +4th or u = 0 to 4, we get

$$\int_{x_0}^{x_0+4h} y'dx = h\int_0^4 \left[y_0^{'} + u\Delta y_0^{'} + \frac{u(u-1)}{2}\Delta^2 y_0^{'} + \frac{u(u-1)(u-2)}{6}\Delta^2 y_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0^1 \right] du$$

(as $dx = h\ du$)

$$\therefore \quad y_4 - y_0 = h[4y_0^{'} + 8\Delta_0^{'} + \frac{20}{3}\Delta^2 y_0^{'} + \frac{8}{3}\Delta^3 y_0^{'} + \frac{28}{90}\Delta^4 y_0^{'}]$$

$$\text{or } y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_2'] + \frac{28}{90}h\Delta^4 y_0' \tag{4.2}$$

This is Milne's predictor formula.

Also integrating (4.1) over the interval x_0 to $x_0 + 2h$ or $u = 0$ to 2,

We get

$$y_2 - y_0 = h(2y_0' + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0' - \frac{1}{20} - \Delta^4 y_0')$$

$$\text{or } y_2 = y_0 + \frac{h}{3}(y_0' + 4y_1 + y_2') - \frac{y}{90}\Delta^4 y_0^1 \tag{4.3}$$

This is Milne's corrector formula.

Since $x_0 \ldots x_4$ *are any five consecutive values of x,* (4.2) and (4.3) can be written in general

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n'), \tag{4.4}$$

$$y_{n+1}^{(1)} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \tag{4.5}$$

It is to be noted that we have considered the difference s up to the third order, because we fit up a polynomial of degree four.

The rerms containing $\Delta^4 y_0'$ are not used directly, but they give the principal parts of the errors in the two values of y_{n+1} computed from (4.4) and (4.5). Note that this error in the (4.5) is of opposite sign to that of (4.4), but it is sufficiently less in magnitude.

As we have taken $\frac{28}{90}h\Delta^4 y'$ and $-\frac{h}{90}\Delta^4 y'$ as the principal parts of the errors, we may take

$$(y_{n+1})_{exact} = y_{n+1} + \frac{28}{90}h\Delta^4 y'$$

$$\text{And } (y_{n+1})_{exact} = y_{n+1}^{(1)} - \frac{h}{90}\Delta^4 y'$$

Where y_{n+1} and $y_{n+1}^{(1)}$ are the predicted and first corrected values of y for $x = x_{n+1}$.

From these two

$$y_{n+1} + \frac{28}{90}h\Delta^4 y' = y_{n+1}^{(1)} - \frac{h}{90}\Delta^4 y'$$

$$\text{or } y_{n+1} - y_{n+1}^{(1)} = 29\left(-\frac{h}{90}\Delta^4 y'\right)$$

$= 29\ (E_2)$.

Where E2 is the principal part of the error in (4.5). Thus

$$E_2 = \frac{1}{29}(y_{n+1} - y_{n+1}^{(1)}).$$

This shows that the error in (3.29) is $\frac{1}{29}$ the of the difference between the predicted and corrected values.

The result helps in determining the error of each computed value or say judging the accuracy of the computed value. If E_2 is small enough, proceed to the next interval. But if E_2 is large enough it means the value of h is large enough and it must be decreased such as by taking its half etc.

Adam's Bashford Method:

To find predictors. Suppose we are interested in finding a formula which uses the information of the function $y(x)$ and its first derivative *i.e.* of $y' = (x)\ f(x, y)$, at the past three points together with one more old value of the derivative. Such a most general linear formula is

$$y_{n+1} = A_0 y_n + A_1 y_{n-1} + A_2 y_{n-2} + h(B_0 y_n + B_1 y'_{n-1} + B_2 y'_{n-2} + B_3 y'_{n-2}) \quad (4.6)$$

It involves seven unknowns. We shall make this formula exact for polynomials upto the degree four. the convenient choices are $y(x) = 1, x, x^2, x^3, x^4$. Putting these values of $y(x)$ respectively in (4.6) and assuming $h = 1$ *i.e.* assuming the unit spacing between the consecutive values of x, we get

$$\left.\begin{aligned} 1 &= A_0 + A_1 + A_2 \\ 1 &= -A_1 - 2A_2 + B_0 + B_1 + B_2 + B_3 \\ 1 &= A_1 + 4A_2 - 2B_1 - 4B_2 - 6B_3 \\ 1 - A_1 &- 8A_2 + 3B_1 + 12B_2 + 27B_3 \\ 1 &= A_1 + 16A_2 - 4B_1 - 32B_2 - 108B_3 \end{aligned}\right] \quad (4.7)$$

These are five equations in 7 unknowns. Taking A_1 and A_2 as parameters and solving (4.7), we get

$$\begin{aligned} A_0 &= 1 - A_1 - A_2 \\ B_0 &= \frac{1}{24}(55 + 9A_1 + 8A_2) \\ B_1 &= \frac{1}{24}(-59 + 19A_1 + 32A_2) \\ B_2 &= \frac{1}{24}(37 - 5A_1 + 8A_2) \\ B_2 &= \frac{1}{24}(-9 + A_1) \end{aligned} \quad (4.8)$$

Here A_1 and A_2 are arbitrary. Taking $A_1 = A_2 = 0$, we get

$$A_0 = 1, B_0 = \frac{55}{24},\ B_1 = -\frac{59}{24}, B_2 = \frac{27}{24}, B_3 = -\frac{9}{24}$$

Putting these values in (3.30), we get

$$y_{n+1} = y_n + \frac{h}{24}[55y_n^{'} - 59y_{n-1}^{'} + 37y_{n-2}^{'} - 9y_{n-2}^{'}] \tag{4.9}$$

This is known as Adam's Predictor formula.

Putting another pairs of values of A_1 and A_2, we can find other predictor formulae. The formulae obtained from (4.9) are called **Adam's Bashford type predictors.**

To find the local truncation error of Adam's predictor *i.e.* of (4.9).

For,

$$y_k = y_0 + (kh)y_0^{'} + \frac{1}{2}(kh)^2 y_0^{(2)} + \frac{(kh)^3}{6}y_0^{(3)} + \frac{(kh)^4}{24}y_0^{(4)} + \frac{(kh)^5}{120}y_0^{(5)} + ...$$

$$\therefore y_{n+1} = y_0 + (n+1)hy_0 + \frac{(n+1)^2h^2}{2}y_0^{(2)} + \frac{(n+1)^3h^3}{6}y_0^{(3)} + \frac{(n+1)^4h^4}{24}y_0^{(4)} + \frac{(n+1)^5h^5}{120}y_0^{(5)} + ...$$

and

$$y_n = y_0 + nhy_0^{'} + \frac{n^2h^2}{2}y_0^{(2)} + \frac{n^3h^3}{6}y_0^{(3)} + \frac{n^4h^4}{24}y_0^{(4)} + \frac{n^5h^5}{120}y_0^{(5)} + ...$$

Also $y_k^{'} = y_0^{'} + (kh).y_0^{(2)} + \frac{(kh)^2}{2}y_0^{(3)} + \frac{(kh)^2}{6}y_0^{(4)} + \frac{(kh)^4}{24}y_0^{(5)} + ...$

$$\therefore y_n^{'} = y_0^{'} + nhy_0^{(2)} + \frac{n^2h^2}{2}y_0^{(3)} + \frac{n^3h^3}{6}y_0^{(4)} + \frac{n^1h^4}{24}y_0^{(5)} + ...$$

$$y_{n-1}^{'} = y_0^{'} + (n-1)hy_0^{(2)} + \frac{(n-1)^2h^2}{2}y_0^{(3)} + \frac{(n-1)^2h^2}{6}y_0^{(4)} + \frac{(n-1)^4h^4}{24}y_0^{(0)} ....$$

$$y_{n-2} = y_0^{'} + (n-2)hy_0^{(2)} + \frac{(n-2)^2}{2}h^2y_0^{(3)} + \frac{(n-2)^2}{6}h^3y_0^{(4)} + \frac{(n-2)^4}{24}h^4y_0^{(3)} + ...$$

$$y'_{n-2} = y'_0 + (n-3)hy_0^{(2)} + \frac{(n-3)^2}{2}h^2 y_0^{(2)} + \frac{(n-3)^3}{6}h^2 y_0^{(4)} + \frac{(n-3)^4}{24}h^4 y_0^{(3)} + \ldots.$$

Putting all these values, we get

$$(y_{n-1} - y_n) - \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}] = \frac{251}{720}h^5 y_0^{(5)} + \ldots.$$

Similar approach may be adapted to the truncation error of the other predictors also.

To find correctors. Like (4.6), the most general linear corrector formula that involves the information about the function and its first derivative at the past three points of the solution, together with an estimate of the derivative at the point being computed, is

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + h(h_{-1}y'_{n+1} + b_0 y'_n + b_1 y'_{n-1} + b_2 y'_{n-2}) \quad (4.10)$$

As usual, making it exact for we get

$$\left.\begin{aligned} a_0 + a_1 + a_2 &= 1 \\ a_1 + 24b_{-1} &= 9 \\ 13a_1 + 8a_2 - 24b_0 &= -19 \\ 13a_1 + 32a_2 - 24b_1 &= 5 \\ a_1 - 8a_2 + 24b_2 &= 1 \end{aligned}\right] \quad (4.11)$$

These are also the five equations in seven unknowns. Taking a_1 and a_2 as parameters, we get

$$a_0 = 1 - a_1 - a_2$$

$$b_{-1} = \frac{1}{24}(9 - a_1)$$

$$b_0 = \frac{1}{24}(19 + 13a_1 + 8a_2)$$

$$b_1 = \frac{1}{24}(-5 + 13a_1 + 13a_2)$$

$$b_2 = \frac{1}{24}(1 - a_1 + 8a_2).$$

The choice $a_1 = a_2 = 0$, which matches to some extent the Adam's predictor, gives

$$y_{n+1} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_{n-1} + y'_{n-2}). \tag{4.12}$$

Various correctors can be found by taking the suitable values of a_1 and a_2.

Proceeding as in the case of (4.9), it can be shown that corrector (4.12) has local truncation error $-\frac{19}{720}h^5 y_0^{(5)}$.

Note that the error in the corrector (4.12) is less enough than that of its predictor (4.9)

The Milne-Type predictors can also be derived by using.

$$y_{n+1} = A_0 y_n + A_1 y_{n-1} + A_2 y_{n-2} + A_3 y_{n-3} + h(b_0 y'_n + B_1 y'_{n-1} + B_2 y'_{n-2}) \tag{4.13}$$

in place of (4.6). The main difference between (4.13), (4.6) is that here we have used an additional sat value of the function i.e.

$$\left.\begin{aligned} A_0 &= -8 - A_2 - 8A_3, B_1 = \frac{14 + 4A_2 - 18A_3}{3} \\ A_1 &= 9 - 9A_3, B_2 = \frac{-1 + A_2 + 9A_3}{3} \end{aligned}\right] \tag{4.14}$$

$$B_0 = \frac{17 + A_2 - 9A_3}{3}.$$

Taking $A_2 = 0$, $A_3 = 1$, we get the Milne's predictor.

1.3 Implementation of multistep method in electrical circuit:

The formation of differential equation for an electric circuit depends upon the following laws.

1. $i = \frac{dq}{dt}$

2. Voltage drop across resistance R=Ri

3. Voltage drop across inductance L= $L\frac{di}{dt}$

4. Voltage drop across capacitance $C = \frac{q}{C}$

Kirchhoff's laws:

- **Voltage law:** The algebraic sum of the voltage drop around any closed circuit is equal to the resultant electromotive force in the circuit.
- **Current law:** At a junction or node, current coming is equal to current going.

(a) L-R series circuit: Let I be the circuit flowing in the circuit containing resistance and inductance L in series, with voltage source E, at any time t.

By voltage law $Ri + L\frac{di}{dt} = E$ or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad (1)$$

This is the differential equation.

IF= $e^{\int \frac{R}{L}dt} = e^{\int \frac{R}{L}t}$

Its solution is $i.e^{\int \frac{R}{L}} = \int \frac{E}{L}e^{\frac{R}{L}t}dt + c$

$$i.e^{\frac{R}{L}t} = \frac{E}{L} \times \frac{L}{R}e^{\frac{R}{L}t} + c \qquad or \qquad i = \frac{E}{R} + ce^{\frac{Rt}{L}} \text{.....(2)}$$

At t=0

The (2) becomes

$$i = \frac{E}{R}[1 - e^{\frac{Rt}{L}}]$$

(b)L-R-C series: Let I be current the circuit containing resistance R, inductance L, and capacitance C in series with voltage source, at any time t.

By voltage law

$$Ri + L\frac{di}{dt} + \frac{q}{c} = E \quad \text{or} \quad R\frac{d^2q}{dt^2} + L\frac{dq}{dt} + \frac{q}{c} = E$$

1.4 Numerical Problem1: A coil having a Resistance of 15 ohms and an Inductance of 10 henries is connected to 90 volts supply. Determine the value of current of after 2 seconds.

Solution-The governing equation may be written as

$$L\frac{dI}{dt} + RI = E$$

So governing equation is

$$10\frac{dI}{dt} + 15I = 90$$

Analytical solution:

$$10\frac{dI}{dt} = 9 - 1.5I$$

Integrating factor will become

$$I = e^{\int 1.5dt}dt$$

$$Ie^{1.5dt} = 9\int e^{1.5t}/1.5 + c_1$$

Since when t=0, I=0 So

$$c_1 = 9/1.5$$

Solution is

$$I = 9/1.5\frac{e^{1.5t}}{e^{1.5t}} - \frac{9}{1.5e^{1.5t}}$$

$$I = 6 - 6e^{-1.5t} \Rightarrow I = 6 - 6\times 0.05$$

at t=2, I=5.7

Taylor series Solution:

Since

$$\frac{dI}{dt}+1.5I=9$$

$$\frac{dI}{dt}=9-1.5I=f(t,I)\Rightarrow(\frac{dI}{dt})_{t=0}=9$$

$$\frac{d^2I}{dt^2}=-1.5\frac{dI}{dt}\Rightarrow(\frac{d^2I}{dt^2})_{t=0}=-1.5\frac{dI}{dt}=-1.5\times9$$

$$\frac{d^3I}{dt^3}=-1.5(\frac{d^2I}{dt^2})=-1.5(\frac{d^3I}{dt^3})_{t=0}=(1.5)^2\times9$$

$$\frac{d^4I}{dt^4}=-1.5\frac{d^3I}{dt^2}\Rightarrow(\frac{d^4I}{dt^4})_{t=0}=-(1.5)^3\times9$$

So

$$I=I_0+tI'_0+\frac{t^2}{2!}I_0''+\frac{t^3}{3!}I_0'''+\frac{t^4}{4!}I_0''''+.....$$

Now putting t=.5 in (1)

I (.5)=3.155275

Now putting t=1

I (1)=4.359375

Now putting t=1.5

I (1.5)=3.2958985

Adam Moltons method:

$$y_0=y(1.5)=3.2958985$$

$$y(1)=y-1=4.359375$$

$$y(1.5)=y-2=3.1555275$$

$$y_0=y_{-3}=0$$

$$y_1^p=y_0+\frac{h}{24}[55f_0-59_{-1}+37f_{-2}-9f_{-3}]$$

$$y_1^p=6.520092042$$

$$y_1^c = y_0 + \frac{h}{24}[9f_1 - 19f_0 - 5f_{-1} + f - 2]$$

$$y_{n+1}^p = 5.513446428$$

Mines method:

$$y_{n+}^c = y_{n-3} + \frac{4h}{3}[2f_{n-1} - f_{n-1} + 2f_n]$$

$$y_{n+1}^c = 5.513446428$$

1.5 Rate of cooling:

Problem2: A body originally at 80^0 C Cools down to 60^0 C in 20 minutes, the temperature of the air being 40^0 C what will be the temperature of the body after 40 minutes from the original.

Analytical solution:

$$\frac{dT}{dt} = -k(T - T_0)$$

Initial condition is when t=0 T=80

$$\frac{dT}{T-40} = -kdt$$

Log(T-40)=-kt+c log40=-c

$$log\left(\frac{t-40}{40}\right) = -kt \qquad (1)$$

t=20 T=60

$$-k \times 20 = \log[\frac{20}{40}] \Rightarrow k = \frac{1}{20} iog2 \qquad (2)$$

By (1)and(2)we have

$$\log\frac{t-40}{40} = (\frac{-1}{20}\log 2)t$$

when t=40

$$\frac{T-40}{40} = -2\log 2 = \log 2^{-2}$$

$$T - 40 = \frac{40}{4} = 10$$

$T = 50\ ^0c$

Numerical solution

T=0 T=80

$$\frac{dT}{dt} = -k(T-40) \qquad \left(\frac{dT}{dt}\right)_0 = -k \times 40$$

$$\frac{d^2T}{dt^2} = -k\frac{dT}{dt} \qquad \left(\frac{d^2T}{dt^2}\right)_0 = -k^2 \times 40$$

$$\left(\frac{d^3T}{dt^3}\right)_0 = -k\frac{d^2T}{dt^2} \qquad \left(\frac{d^3T}{dt^3}\right)_0 = -k^2 \times 40$$

$$\frac{d^4T}{dt^4} = -k\frac{d^3T}{dt^3} \qquad \left(\frac{d^4T}{dt^4}\right)_0 = k^4 \times 40$$

$$\frac{dT}{dt} = -k(T-40)$$

t=0 T=40

$$T(t) = T + tT^1 + \frac{t^2T^{"}}{2!} + \frac{t^3T'''}{3!} + \frac{t^4T^{iv}}{4!} + ...$$

$$T(30) = 80 + 36(-40k) + \frac{36^2}{2!}(40k^2) + \frac{36^3}{3!}(-40k^3) + \frac{36^4}{4!}(40k^4)$$

$$T(37) = 80 + 37(-40k) + \frac{37^2}{2!}(40k^2) + \frac{37^3}{3!}(-40k^3) + \frac{37^4}{4!}(40k^4)$$

$$T(38) = 80 + 38(-40k) + \frac{38^2}{2!}(40k^2) + \frac{38^3}{3!}(-40k^3) + \frac{38^4}{4!}(40k^4)$$

$$T(39) = 80 + 39(-40k) + \frac{39^2}{\sqrt{2}}(-40k^2) + \frac{39^3}{\sqrt{3}}(-40k^3) + \frac{39^4}{\sqrt{4}}(-40k^4) + \frac{39^5}{\sqrt{5}}(-40k^5).....$$

Since K=0.03466

T(36)=30.0896+31.13810-12.95096+4.03992

T(36)=52.3166

T(37)=28.7032+32.89202-14.0605+4.5079

T(37)=52.04262

T(38)=27.3168+34.69399-15.2316+5.015305

T(38)=51.79424

T(39)=25.9304+36.5440-16.46600+5.5644

T(39)=51.57284

T	Say	T	F=-K(T-40), K=0.03466
36	f_{-3}	**52.3166**	-0.42696
37	f_{-2}	**52.04262**	-0.417397
38	f_{-1}	**51.79424**	-0.408788
39	f_0	**51.57284**	-0.40115

Adams-Moulton method:

$$y_1^p = y_0 + \frac{h}{24}\left[55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right]$$

So y_1^p =51.1751

$$y_1^c = y_0 + \frac{h}{24}\left[9 f_1^p + 19 f_0 - 5 f_{-1} + f_{-2}\right]$$

y_1^c =51.177786

Milne'S Method:

$$y_{n+1}^p = y_{n-3} + \frac{4h}{3}\left[2 f_{n-2} - f_{n-1} + 2 f_n\right]$$

y_1^p =51.0883643

$$y_{n+1}^c = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4 f_n + f_{n-1}\right]$$

y_1^c =50.976

1.6 Conclusion:

In this section we are given the comparison of numerical solution with analytical solution

Adams-Moulton method:	**Milne's Method:**	**Analytical method**
$y_{n+1}^{p} = 5.513446428$	$y_{n+1}^{c} = 5.513446428$	I=5.7

Comparison of multistep method by Analytical solution for Numerical problem 1

Adams-Moulton method:	**Milne'S Method:**	**Analytical method**
y_1^c =51.177786	y_1^c =50.976	50

Comparison of multistep method by Analytical solution for Numerical problem 2

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