EUROPEAN ACADEMIC RESEARCH
Vol. II, Issue 1/ April 2014

Impact Factor: 3.1 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Finite Element Approximation for One Dimensional Parabolic Problem

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Abstract:

In this paper, we have studied the finite element approximation for 1D parabolic problems. We also discussed about the convergence of finite element solution to the exact solution of parabolic problem with respect to L^2 norm. The main objective of this paper is to introduce the finite element method for 1D parabolic problem. The paper discussed the finite element method for continuous time discretization and discrete in time discretization schemes. Finally, the convergence of finite element solution to exact solution is discussed in L^2 norm.

Key words: Finite Element, L^2 norm.

Introduction:

The mathematical models of science and engineering mainly take the form of differential equations. Most of the real problems are defined on domains that are geometrically complex and many have different boundary conditions on different portions of the boundary. Therefore, it is usually impossible (or difficult) to solve a solution analytically and in general one has to rely on numerical techniques finding approximate solutions. One such numerical technique is finite element method.

Finite element method is a rigorous method for solving partial differential equations. It has a much stronger mathematical foundation than many other methods (It has a more elaborate mathematical foundation than many other methods) and particularly finite difference method.

The finite element method is generally a powerful computational technique for the solution of differential and integral equations that arise in various fields of engineering and applied sciences and mathematically a generalization of the classical variational (Ritz) and weighted-residual (Galerkin, least-squares, etc).

In finite element method, we seek the approximate solution in a finite number of sub-domains instead of solving the problem in the whole computational domain. Such sub-domain are called elements. The collection of all elements are known as mesh. If the elements are of same size then it is known as uniform mesh otherwise it is called non-uniform mesh.

The main objective of this paper is to study the finite element method for two-dimensional parabolic initial boundary value problems. More precisely, the convergence of finite element solution to the exact solution has studied at time t with respect to L^2 norm and H^1 norm.

Basic Definitions:

Definition1. Let $\Omega \subseteq \mathbb{R}$ and p is a real number with the property $1 \leq p < \infty$, then $L^p(\Omega)$ denotes the following space

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} : (\int |f(x)|^p \, dx)^{\frac{1}{p}} < \infty\}.$$

Further, $L^p(\Omega)$ is a normed linear space with respect to the following norm

$$\|f\|_{L^p(\Omega)} = \{(\int_{\Omega} |f(\mathrm{x})|^p\, dx)^{\frac{1}{p}}\}$$

Definition2. In order to introduce the weak derivative, we consider the following

equation $\frac{dy}{dx}$ = g(x) in Ω

It is known that if g ∈ c (Ω) then y∈ C^1 (Ω). The problem arises if g ∈ L^2(Ω), then we cannot expect the solution in C^1(Ω). Naturally, we assume that y ∈ C(Ω) and need not be differentiable. Thus, we cannot find $\frac{dy}{dx}$ in classical sense. Therefore we need to attach a meaning to the derivative of y even y is not differentiable or not at all continuous. This is done by introducing weak derivative of such functions.

To motivate towards weak derivative of such functions let us try to define the weak derivative of a differentiable function f and then we will generalize the same definition for larger class of functions. Let $C_0^\infty(\Omega)$ be the collection of all C^∞ functions define over Ω which vanishes on the boundary $\partial\Omega$ of Ω. Then, for v∈ $C_0^\infty(\Omega)$,we have

$$\int_\Omega \frac{df}{dx} v\, dx = fv]_{\partial\Omega} - \int_\Omega f \frac{dv}{dx}\, dx$$

$$= - \int_\Omega f \frac{dv}{dx}\, dx.$$

Thus, under the sign of integration, derivative of f reduces to the derivative of v. Therefore the first order weak derivative of f given by D_f: $C_0^\infty(\Omega) \to \mathbb{R}$ such that

$$D_f(\mathrm{v}) = -\int_\Omega f v' dx$$

Since this does not involve the derivative of f, we can define the weak derivative for those functions f, which not at all differentiable or continuous. In general, the m th order weak derivative of f is defined as

$$D_f^m(\mathrm{v}) = (-1)^m \int_\Omega v^{(m)} f\, dx.$$

Definition3. Let m>0 be an integer and let $1 \leq p < \infty$. Then the sobolev space $W^{m,p}(\Omega)$ is defined as

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega): D^a \in L^P(\Omega) \ \forall \ |a| \leq m\}$$

Where D^a is the a th order weak derivative of u.

Definition4. For p=2 the sobolev space $W^{m,p}(\Omega)$ is a Hilbert space and it is denoted by $H^m(\Omega)$.

In particular

$H^1(\Omega)=\{v \in L^2(\Omega) : Dv \in L^2(\Omega)\}$

And the corresponding norm is defined

$$\| v \|_{H^1(\Omega)} = \| v \|_{L^2(\Omega)} + \| Dv \|_{L^2(\Omega)}$$

Definition5. The collection of all H^1 functions vanishing on the boundary is a closed subspace of

$H^1(\Omega)$ and it is denoted by $H_0^1(\Omega)$ and defined as

$H_0^1(\Omega)$

$$H_0^1(\Omega)=\{v \in \{H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

Finite Element Approximation for 1D Parabolic Problem:

Let us consider the non-stationary heat conduction problem in a rod $\Omega = (a, b)$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \ , \quad a<x<b, \ t>0 \qquad (1.0.1)$$

With the boundary condition u(a, t)=u(b, t)=0, t>0 and initial condition u(x,0)= $u_0(x)$, a<x<b.

The finite element method follows the following basic steps:

Step1: Weak Formulation: In this step we reduce the order of the differential equation.

Here, the weak formulation of the above problem is: Find

$u(\mathrm{t}) \in \mathrm{H}_0^1(\Omega), \mathrm{t} \in \mathrm{I} = (0, \mathrm{T})$

(Where T is given time) such that

$$(\dot{u}(\mathrm{t}), \mathrm{v}) + \mathrm{a}(\mathrm{u}(\mathrm{t}), \mathrm{v}) = (\mathrm{f}(\mathrm{t}), \mathrm{v}) \forall \mathrm{v} \in \mathrm{H}_0^1(\Omega), \mathrm{t} \in \mathrm{I} \qquad (1.1.1)$$

With u(0)= u_0 and a(.,.) is the bilinear map associated with this problem and (.,.) is the standard L^2 inner product.

Step 2: Semi-discretization: Let (a, b) be the given domain. Then in this step we discretize the domain (a, b) by the points $x_0 = a, x_1 = x_0 + h, ..., x_n = b = x_0 + nh$ with $h = \dfrac{b-a}{n}$, which is known as mesh parameter.

Step3: Construction of finite element space:
In this step, we construct a piecewise linear finite element space V_h with respect to the mesh parameter h. To do so, we construct basis functions $\Phi_1, \Phi_2, ..., \Phi_{n-1}$ corresponding to the grid points $x_1, x_2, ..., x_{n-1}$ such that

$$\Phi_i(\mathrm{x}_j) = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

With the property that each Φ_i is piecewise linear function and the set $\{\Phi_1, \Phi_2, ..., \Phi_{n-1}\}$ is linearly independent. One obvious choice for Φ_i is

$$\Phi_i(\mathrm{x}) = \frac{x - x_{i-1}}{h} \text{ if } x_{i-1} \le x \le x_i$$

x$_{\text{i-1}}$ £ x £ x$_{\text{i}}$

$$= \frac{x_{i+1} - x}{h} \text{ if } x_i \le x \le x_{i+1}$$
$$= 0 \text{ else}$$

Then, we define V h = span $\{\Phi_1, \Phi_2, ..., \Phi_{n-1}\}$ to be the finite element space.

4. Step 4: Finite element approximation:
The finite element approximation of u(t) is defined as: Find $u_h(t) \in V_h, t \in I$ such that

$$(\dot{u}_h(t), v_h) + a(u_h(t), v_h) = (f(t), v_h) \forall v_h \in V_{h,} t \in I \qquad (1.2.1)$$

With $u_h(0)$ is a suitable approximation to u_0.

Since $u_h \in V_h = \text{span}\{\Phi_1, \Phi_2, ..., \Phi_{n-1}\}$. Therefore there exist unique time dependent coefficients $c_1(t), c_2(t), ..., c_{n-1}(t)$ such that

$$u_h(t) = c_1(t)\Phi_1 + c_2(t)\Phi_2 + ... + c_{n-1}(t)\Phi_{n-1}.$$

Then, we substitute this value of $u_h(t)$ in (1.2.1) and setting $v_h = \Phi_1, \Phi_2, ..., \Phi_{n-1}$ we get a system of n-1 equations. In matrix form which can be written as

$$B\dot{C}(t) + AC(t) = F(t) \forall t \in I \qquad (1.3.1)$$

With $B = (b_{ij}), A = (a_{ij}), F = (f_i), C = (c_i), u_0 = u_0(i).$ Further, (1.3.1) can be written as

$$\dot{C}(t) + \bar{A}C(t) = g(t), t \in I \qquad (1.4.1)$$

With $C(0) = C_0$ and $\bar{A} = B^{-1}A, g = B^{-1}F$.
Equation (1.4.1) is a system of first order linear differential equation with continuous coefficient matrices. So equation (1.4.1) has a unique solution. The solution is given by

$$C(t) = e^{-\bar{A}t} + \int_0^t e^{\bar{A}s} g(s)\, ds, t \in I.$$

Error Analysis:

The weak form of the given differential equation (1.0.1) is

$$(\dot{u}(t), v) + a(u(t), v) = (f(t), v) \forall v \in H_0^1(\Omega), t \in I$$

And finite element approximation is

$$(\dot{u}_h(t), v_h) + a(u_h(t), v_h) = (f(t), v_h) \forall v_h \in V_{h,} t \in I$$

With $u_h(0)$ is a suitable approximation to u_0. In order to study the convergence, we consider a projection $p_h : H_0^1(\Omega) \to V_h$ such that

$$(p_h u, v_h) = (u, v_h) \forall v_h \in V_h.$$

We then write the error $e = u - u_h$ as

$$e = u - p_h u + p_h u - u_h = \rho + \theta$$

Where $\rho = u - p_h u$ and $\theta = p_h u - u_h$. Again ρ satisfies an error of the form

$$\|\rho(t)\|_{L^2(\Omega)} = \|u - p_h u\|_{L^2(\Omega)} \le ch^2 \|u\|_{H^2(\Omega)}$$

In order to bound θ, we note that

$$\begin{aligned}
(\theta_t, X) + a(\theta, X) &= ((p_h u - u_h)_t, X) + a(p_h u - u_h, X) \\
&= (p_h u_t, X) + a(p_h u, X) - (u_{ht}, X) - a(u_h, X) \\
&= (p_h u_t, X) + a(u, X) - (f, X) \\
&= (p_h u_t, X) + a(u, X) - (u_{t,} X) - a(u, X) \\
&= (p_h u_t - u_t, X) \\
&= (\rho_{t,} X) \forall X \in V_{h.}
\end{aligned}$$

Then, we set X= θ to get the following estimate

$$\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \|\rho_t(t)\| ds$$

This together with the estimate for ρ leads to the following result

$$\|u-u_h\|_{L^2(\Omega)} \le \|\rho\| + \|\theta\|$$
$$\le ch^2\|u\|_{H^2(\Omega)} + ch^2\|u_t\|_{H^2(\Omega)}$$

Thus, we have the following L^2 norm error estimate:

Theorem: Let $f \in L^2(\Omega)$ and $u_0 \in H_0^1(\Omega)$ then we have

$$\|u-u_h\|_{L^2(\Omega)} \le ch^2(\|u\|_{H^2(\Omega)} + \|u_t\|_{H^2(\Omega)})$$

Discrete in Space and Time Scheme for Parabolic Problem:

The Backward Euler Method:

We discretize the time interval I=(0, T) by the points $t_0, t_1, \ldots\ldots, \mathrm{t}_N$ such that $0 = t_0 < t_1 < \ldots\ldots < \mathrm{t}_N = T$ and write $I_n = (\mathrm{t}_{n-1}, \mathrm{t}_n)$ and let $k = \mathrm{t}_n - \mathrm{t}_{n-1}, n = 0,1,\ldots,N$ be the local time step. Let u_h^n be the approximation in V_h of u(t) at $t = t_n = t_0 + nk = nk$. Then the Backward Euler method is defined by replacing time derivative in semi-discrete problem by backward difference quotient $\frac{u_h^n - u_h^{n-1}}{k}$.

Therefore, fully discrete scheme is defined as: Find $u_h^n \in V_h, n = 1,2,\ldots,N$ such that $(\frac{u_h^n - u_h^{n-1}}{k}, \mathrm{v}_h) + \mathrm{a}(\mathrm{u}_h^n, \mathrm{v}_h) = (\mathrm{f}(\mathrm{t}_n), \mathrm{v}_h) \forall\, \mathrm{v}_h \in V_h$

With the initial condition $(\mathrm{u}_h^0, \mathrm{v}_h) = (\mathrm{u}^0, \mathrm{v}_h) \forall\, \mathrm{v}_h \in V_h$

Thus $(\mathrm{u}_h^n, \mathrm{v}_h) - (\mathrm{u}_h^{n-1}, \mathrm{v}_h) + \mathrm{ka}(\mathrm{u}_h^n, \mathrm{v}_h) = \mathrm{k}(\mathrm{f}(\mathrm{t}_n), \mathrm{v}_h) \qquad (2.0.1)$

Since $u_h^n \in V_h = span\{\Phi_1, \Phi_2, ..., \Phi_{n-1}\}$. So there exists $c_1^n, c_2^n, ..., c_{n-1}^n \in$ such that

$$u_h^n = \sum_{i=1}^{n-1} c_i^n \Phi_i(\mathrm{x})$$

Therefore (2.0.1) gives

$$(\sum_{i=1}^{n-1} c_i^n \Phi_i, v_h) - (\sum_{i=1}^{n-1} c_i^{n-1} \Phi_i, v_h) + \mathrm{ka}(\sum_{i=1}^{n-1} c_i^n \Phi_i, v_h) = \mathrm{k}(\mathrm{f}(\mathrm{t}_n), \mathrm{v}_h) \quad (2.1.1)$$

Writing $v_h = \Phi_j, j = 1, 2, ..., n-1$ in (2.1.1), we have

$$(\sum_{i=1}^{n-1} c_i^n \Phi_i, \Phi_j) - (\sum_{i=1}^{n-1} c_i^{n-1} \Phi_i, \Phi_j) + \mathrm{ka}(\sum_{i=1}^{n-1} c_i^n \Phi_i, \Phi_j) = \mathrm{k}(\mathrm{f}(\mathrm{t}_n), \Phi_j)$$

$$\Rightarrow \sum_{i=1}^{n-1} c_i^n (\Phi_i, \Phi_j) - \sum_{i=1}^{n-1} c_i^{n-1} (\Phi_i, \Phi_j) + \mathrm{k} \sum_{i=1}^{n-1} c_i^n a(\Phi_i, \Phi_j) = \mathrm{k}(\mathrm{f}(\mathrm{t}_n), \Phi_j)$$

It can be written as

$$C^n B - C^{n-1} B + kC^n A = kF(\mathrm{t}_n)$$

$$\Rightarrow (\mathrm{B} + \mathrm{kA})\mathrm{C}^n = BC^{n-1} + kF(\mathrm{t}_n)$$

Where

$C^n = c_i^n, B = (\Phi_i, \Phi_j), \mathrm{A} = \mathrm{a}(\Phi_i, \Phi_j), \mathrm{F}(\mathrm{t}_n) = (\mathrm{f}(\mathrm{t}_n), \Phi_j)$.

Further B+ kA is positive definite and hence invertible. Therefore, we get unique u_h^n.

Regarding the stability of u_h^n, we have

$$\|u_h^n\|_{L^2(\Omega)} \leq c\{\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\}$$

And this can be proved by setting $v_h = u_h^n$ in (2.0.1)

Error Analysis:

Let u_h^n be the finite element solution of (2.0.1) at t=t_n and u^n be the exact solution of (2.0.1) at t=t_n. Let us consider a projection $R_h : L^2(\Omega) \to \mathrm{V}_h$ as

$$(\mathrm{R}_h \mathrm{u}^n, \mathrm{v}_h) = (\mathrm{u}^n, \mathrm{v}_h) \forall \mathrm{v}_h \in V_h .$$

We then write the error

$$e = u_h^n - u^n = (\mathrm{u}_h^n - \mathrm{R}_h \mathrm{u}^n) + (\mathrm{R}_h \mathrm{u}^n - \mathrm{u}^n) = \theta^n + \rho^n$$

And $\rho^n = \rho(\mathrm{t}_n)$ is bounded as

$$\left\|\rho^n\right\|_{L^2(\Omega)} \le ch^2 \left\|u^n\right\|_{L^2(\Omega)} .$$

In order to bound θ, we note that

$$(\frac{\theta^n - \theta^{n-1}}{k}, \mathrm{X}) + \mathrm{a}(\theta^n, \mathrm{X}) =$$

$$(\frac{u_h^n - R_h u^n - u_h^{n-1} + R_h u^{n-1}}{k}, X) + a(\mathrm{u}_h^n - \mathrm{R}_h \mathrm{u}^n, \mathrm{X})$$

$$= -(\frac{R_h(\mathrm{u}^n - \mathrm{u}^{n-1})}{k}, \mathrm{X}) + (\frac{u_h^n - u_h^{n-1}}{k}, \mathrm{X}) + \mathrm{a}(\mathrm{u}_h^n, \mathrm{X}) - \mathrm{a}(\mathrm{R}_h \mathrm{u}^n, \mathrm{X})$$

$$= -(\frac{R_h(\mathrm{u}^n - \mathrm{u}^{n-1})}{k}, \mathrm{X}) + (\mathrm{f}(\mathrm{t}_n), \mathrm{X}) - \mathrm{a}(\mathrm{R}_h \mathrm{u}^n, \mathrm{X})$$

$$= -(\frac{R_h(\mathrm{u}^n - \mathrm{u}^{n-1})}{k}, \mathrm{X}) - \mathrm{a}(\mathrm{R}_h \mathrm{u}^n, \mathrm{X}) + (\mathrm{u}_t^n, \mathrm{X}) + \mathrm{a}(\mathrm{u}^n, \mathrm{X})$$

$$= -(\frac{R_h(\mathrm{u}^n - \mathrm{u}^{n-1})}{k}, \mathrm{X}) - \mathrm{a}(\mathrm{u}^n, \mathrm{X}) + (\mathrm{u}_t^n, \mathrm{X}) + \mathrm{a}(\mathrm{u}^n, \mathrm{X})$$

$$= -(\frac{R_h(\mathrm{u}^n - \mathrm{u}^{n-1})}{k}, \mathrm{X}) + (\mathrm{u}_t^n, \mathrm{X})$$

$$= -(\frac{R_h(\mathrm{u}^n - \mathrm{u}^{n-1})}{k} - \mathrm{u}_t^n, \mathrm{X})$$

$$= -(\mathrm{w}^n, \mathrm{X}), \forall\, \mathrm{X} \in \mathrm{V}_h, n \geq 1$$

Where $w^n = R_h(\frac{u^n - u^{n-1}}{k}) - \mathrm{u}_t^n$

$$= (\mathrm{R}_h - \mathrm{I})\frac{u^n - u^{n-1}}{k} + (\frac{u^n - u^{n-1}}{k} - \mathrm{u}_t^n)$$

$$= w_1^n + w_2^n$$

Choosing $X = \theta^n$, we have

$$(\frac{\theta^n - \theta^{n-1}}{k}, \theta^n) + \mathrm{a}(\theta^n, \theta^n) = -(\mathrm{w}^n, \theta^n)$$

Or, $$(\frac{\theta^n - \theta^{n-1}}{k}, \theta^n) \leq -(\mathrm{w}^n, \theta^n)$$

$$\leq \left\|(\mathrm{w}^n, \theta^n)\right\|$$

$$\leq \left\|\mathrm{w}^n\right\|\left\|\theta^n\right\|$$

Or, $$(\theta^n, \theta^n) - (\theta^{n-1}, \theta^n) \leq k\left\|\mathrm{w}^n\right\|\left\|\theta^n\right\|$$

Or, $$\left\|\theta^n\right\|^2 - (\theta^{n-1}, \theta^n) \leq k\left\|\mathrm{w}^n\right\|\left\|\theta^n\right\|$$

Or, $$\left\|\theta^n\right\|^2 \leq k\left\|\mathrm{w}^n\right\|\left\|\theta^n\right\| + (\theta^{n-1}, \theta^n)$$

$$\leq (k\left\|\mathrm{w}^n\right\| + \left\|\theta^{n-1}\right\|)\left\|\theta^n\right\|$$

Or, $$\left\|\theta^n\right\| \leq \left\|\theta^{n-1}\right\| + k\left\|\mathrm{w}^n\right\|$$

And by repeated application,

$$\left\|\theta^n\right\| \le \left\|\theta^0\right\| + k\sum_{j=1}^{n}\left\|w^j\right\|$$

$$\le \left\|\theta^0\right\| + k\sum_{j=1}^{n}\left\|w_1^j\right\| + k\sum_{j=1}^{n}\left\|w_2^j\right\|$$

And

$$\left\|\theta^0\right\| = \left\|u_h^0 - R_h u^0\right\|$$

Taking

$$R_h u^0 = u_h^0,$$

We have $\left\|\theta^0\right\| = 0$.

Further,

$$w_1^j = (\mathrm{R}_h - \mathrm{I})\frac{u^j - u^{j-1}}{k}$$

$$= (\mathrm{R}_h - \mathrm{I})\,\mathrm{k}^{-1}\int_{t_{j-1}}^{t_j} u_t\,ds$$

$$= \mathrm{k}^{-1}\int_{t_{j-1}}^{t_j} (\mathrm{R}_h - \mathrm{I})u_t\,ds$$

And we obtain

$$k\sum_{j=1}^{n}\left\|w_1^j\right\| \le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|(\mathrm{R}_h - \mathrm{I})\,\mathrm{u}_t\right\|ds$$

$$\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} ch^2\left\|u_t\right\|_{L^2(\Omega)}ds$$

$$= ch^2\int_0^{t_n}\left\|u_t\right\|_{L^2(\Omega)}ds$$

Similarly, for w_2^j, we have

$$w_2^j = \frac{u^j - u^{j-1}}{k} - u_t^j$$

$$\Rightarrow kw_2^j = u^j - u^{j-1} - ku_t^j$$

$$= -\int_{t_{j-1}}^{t_j} (\mathrm{s} - \mathrm{t}_{j-1})\, \mathrm{u}_{tt}(\mathrm{s})\, \mathrm{ds}$$

So that

$$k\sum_{j=1}^{n} \left\| w_2^j \right\| \le \sum_{j=1}^{n} \left\| \int_{t_{j-1}}^{t_j} (\mathrm{s} - \mathrm{t}_{j-1})\, \mathrm{u}_{tt}(\mathrm{s})\, \mathrm{ds} \right\|$$

$$\le k\int_0^{t_n} \left\| u_{tt} \right\| ds$$

Therefore, $\left\| u_h^n - u^n \right\|_{L^2(\Omega)} = \left\| \theta^n + \rho^n \right\|$

$$\le \left\| \rho^n \right\| + \left\| \theta^n \right\|$$

$$\le ch^2 \left\| u^n \right\|_{L^2(\Omega)} + ch^2 \int_0^{t_n} \left\| u_t \right\|_{L^2(\Omega)} ds + k\int_0^{t_n} \left\| u_{tt} \right\| ds$$

Thus we have the following L^2 norm error estimate:

Theorem: If u_h^n be the finite element solution at $t = t_n$ and u^n be the exact solution at $t = t_n$

Then, we have

$$\left\|u_h^n - u^n\right\|_{L^2(\Omega)} \le ch^2 \left\|u^n\right\|_{L^2(\Omega)} + ch^2 \int_0^{t_n} \left\|u_t\right\|_{L^2(\Omega)} ds + k \int_0^{t_n} \left\|u_{tt}\right\| ds$$

Example: We consider the following initial boundary value problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(\mathrm{x}, \mathrm{t}), \mathrm{x} \in (0,1), \mathrm{t} \in (0,1)$$

With boundary condition

$u(0,\mathrm{t}) = \mathrm{u}(1,\mathrm{t}) = 0 \forall\, \mathrm{t}$

And initial condition

$u(\mathrm{x},0) = u_0(\mathrm{x})$

Let $v(\mathrm{x}) = \frac{x}{4} - \frac{x^2}{4}$. Then we select f(x,t) in such a way $u(\mathrm{x},\mathrm{t}) = \mathrm{v}(\mathrm{x})\,\mathrm{e}^{\sin(\mathrm{t})}$ is a solution.

Therefore $f(\mathrm{x},\mathrm{t}) = v(x)\cos(\mathrm{t})\,\mathrm{e}^{\sin(\mathrm{t})} + \frac{1}{2}$.

At t=1/2, we have the following errors for different h.

H	$\left\|u - u_h^n\right\|_{L^2(\Omega)}$	ratio V_h
1/2	.00024	
1/4	.00006	4
1/8	.000015	4

Conclusion:

In this paper, we illustrated numerical solutions of parabolic equations using finite Element method. Finite element method is most frequently applied numerical approximations, although several numerical methods are available. We illustrated how finite element method utilizes discrete elements to obtain the approximate solutions of the governing differential equation.

Finite element method is evolving with technology. The growth in computer technology has made it even more possible to consider using them in many science and engineering applications. Thus, this paper intended to give some fundamental introduction into finite element method by considering simple example.

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