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Mathematical Aspects of a Vibrating String

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Abstract

A wave can be understood as a disturbance that propagates in a medium. There is a wide variety of waves in nature, and the study of their properties and behavior is an important field of physics. Among the most fundamental properties associated with a wave is the fact that it carries energy without dragging the material medium where it propagates. The purpose of this paper is to make a mathematical review of the propagation of a transverse mechanical wave through a string. The present work aims to study a possible

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analytical solution for the Wave Equation, where we approach the problem of the vibrating string. This equation is written as a Partial Differential Equation (PDE) of the second order.

Key words: vibrating string, wave, differential equation

INTRODUCTION

A wave occurs [(1)] when a system is displaced from its equilibrium position and the disturbance can move or propagate from one region to another of the system. The waves are relevant in several branches of physical and biological science such as: string oscillation, sound, light, sea waves, radio and television transmission and earthquakes. Mechanical Waves [(1)] are the most familiar since they are associated with water, sound, seismic shocks, etc. All have certain central characteristics, for they are governed by Newton's laws and can exist only within a material medium, as in water, air, rocks, etc. The analysis carried out in this text follows the conventional formulations of classical mechanics. So our main interest is to do a review of the mathematics involved in the movement of a theoretical string, which is attached to its two ends at fixed points and which has been displaced from its equilibrium by an external agent.

The vibrating string [(1)] problem aroused great interest within the scientific community in vibration problems of continuous systems. The first to deduce the equation of the vibrating string considered as a continuous medium was Jean Le Rond d'Alembert in 1756. A century later, in his remarkable work "The Theory of Sound", Lord Rayleigh makes a very complete treatment involving continuous and discrete vibration systems and of course deals with the vibrating string problem.

Considering that a string of a musical instrument is stretched, subjected to a certain tension and attached to the ends, it will then perform free oscillations, ie without damping, on its equilibrium position, subject only to the action of the restoring force, which tends to bring the rope back to the equilibrium position.

MATHEMATICAL ANALYSIS

Suppose (2,3) the string has length L and its ends are fixed at points x = 0 and x = L. Let us also assume that the string has a uniform linear density, that is, mass per unit of length is constant, given by $\mu = \frac{dm}{dx}$, and which is stretched with a constant tension T, as shown in figure 1.

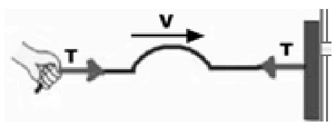

Figure 1- String with uniform linear density stretched with a constant tension T

Source: https://www.google.com.br/search?cordavibranteimagens&source

At some point in time, a piece of rope will be in the generic position indicated by figure 2.

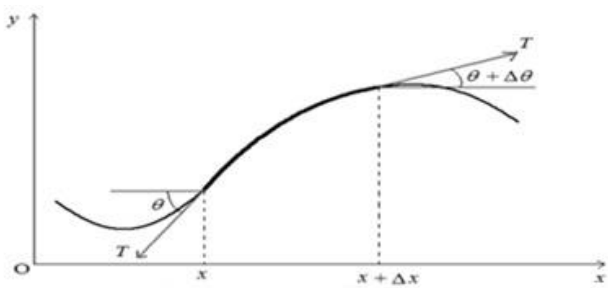

Figure 2 - Illustration of the rope position at time t

Source: https://www.google.com.br/search?cordavibranteimagens&source

On the other hand, the mass of a small rope segment of length Δx, highlighted in the figure is

$$\Delta m = \mu \Delta x \tag{1}$$

The horizontal and vertical components of the resultant force acting on this string segment are

$$F_x = T\cos\,(\theta + \Delta\theta) - T\cos\theta \qquad (2)$$

and

$$F_y = T\sin\,(\theta + \Delta\theta) - T\sin\theta \qquad (3)$$

Let us now assume [2,3] that the string does not move in the x-direction, that is, the string motion occurs only on the y-axis. This means that the resulting force in the x direction is zero, this is, $F_x = 0$. Substituting this conclusion into equation (2) we have

$$\cos\,(\theta + \Delta\theta) = \cos\theta \qquad (4)$$

By Newton's second law we can obtain the resultant force in the direction y

$$\mathrm{F_y} = (\mu\,\Delta x)a_y = (\mu\,\Delta x)\,\frac{\partial^2 y}{\partial t^2} \qquad (5)$$

In equation (5) we have that the mass is given by $\Delta m = \mu\Delta x$ and the acceleration in the y direction is given by $\frac{\partial^2 y}{\partial t^2}$. Substituting equation (5) into equation (3) comes that

$$T\sin\,(\theta + \Delta\theta) - T\sin\theta = (\mu\,\Delta x)\,\frac{\partial^2 y}{\partial t^2} \qquad (6.1)$$

or

$$\sin\,(\theta + \Delta\theta) - \sin\theta = \Delta x\,\frac{\mu}{T}\,\frac{\partial^2 y}{\partial t^2} \qquad (6.2)$$

Now let's divide the two sides of equation (6.2) by the term cos θ, but this will be done on the basis of equation (4), which says that

$$\cos\,(\theta + \Delta\theta) = \cos\theta$$

Thus the term sin (θ + Δθ) will be divided by cos(θ + Δθ) and the term sin θ will be divided by cos θ and the right side of the equation will also be divided by the term cos θ. In this way we will have

$$\frac{\sin\,(\theta+\Delta\theta)}{\cos\,(\theta+\Delta\theta)} - \frac{\sin\theta}{\cos\theta} = \frac{\Delta x}{\cos\theta}\,\frac{\mu}{T}\,\frac{\partial^2 y}{\partial t^2} \qquad (7.1)$$

This equation implies that

$$\tan\,(\theta + \Delta\theta) - \tan\theta = \frac{\Delta x}{\cos\theta}\,\frac{\mu}{T}\,\frac{\partial^2 y}{\partial t^2} \qquad (7.2)$$

Taking into account [2,3] that the angular coefficient of the line tangent to a function at a given point in its domain is the same as the derivative of the function at this point, then equation (7.2) can also be written in terms of partial derivatives given by

$$\frac{\partial y(x+\Delta x,t)}{\partial x} - \frac{\partial y(x,t)}{\partial x} = \frac{\Delta x}{\cos\theta}\,\frac{\mu}{T}\,\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (8)$$

If we divide the two sides of equality (8) by Δx we will have in the left side the term

$$\frac{\dfrac{\partial y(x+\Delta x,t)}{\partial x} - \dfrac{\partial y(x,t)}{\partial x}}{\Delta x}$$

At the limit on what Δx tends to zero, this equation becomes the partial derivative with respect to x of the term $\frac{\partial y}{\partial x}$ which, in turn, is the second partial derivative $\frac{\partial^2 y}{\partial x^2}$. Then equation (8) will be written

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{\cos\theta}\,\frac{\mu}{T}\,\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (9)$$

Let us now consider [2,3] that the strings displacements are small. This assumption implies that the angles associated with these displacements are also very small, that is, the angles tend to zero. Considering this condition, $cos\theta \approx 1$, equation (9) becomes

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\mu}{T}\,\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (10)$$

This is the so-called vibrating string equation that was first published in 1747 by French philosopher and mathematician Jean d'Alembert. The term μ/T has a dimension of $1/(\text{velocity})^2$, so it is usual to write equation (10) as follows

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (11)$$

Then v is identified with the wave propagation velocity in the stretched string

$$v = \left(\frac{T}{\mu}\right)^{\frac{1}{2}} \qquad (12)$$

An important observation is that Equation (11) is not only valid for transverse waves in a stretched string. In fact, it holds for any wave in one dimension. Equation (11) is called a one-dimensional wave equation and equation (10) is just one particular case of it. The wave equation (for one dimension or more) is one of the most important equations in physics in the study of wave phenomena.

FROM VIBRATING STRINGS TO THE STRUCTURE THEORY

In the last 40 years the String Theory (4) has developed as the best candidate for a unified theory of Nature. It assumes that all the particles are different harmonics of small vibrating strings, similar to how the different harmonies of a violin string correspond to different musical notes. This simple premise leads to exciting implications in solving the problem of reconciling Einstein's theory of gravitation with the laws of quantum mechanics. According to string theory (4), the universe is composed not of spot particles, but of small vibrating strings, and the differences in vibrations of such strings constitute the different behaviors of matter.

String theory was originally created to explain the peculiarities of hadrons behavior. The physicist Gabriele Veneziano, in 1968, trying to discover the meaning of some properties of the strong nuclear force, realized that a formula of the mathematician Leonhard Euler could describe all the properties of the particles that have strong interaction. With this finding, a large amount of research on the application of Euler's beta function to the strong interaction particles was made. However, no one knew why the formula worked. In 1970, Yoichiro Nambu and Holger Nielsen showed (4) that if the elementary particles were formed from tiny, one-dimensional vibrating strings, their interactions could be described exactly by Euler's function.

In experiments on particle accelerators, physicists have observed that the angular momentum of a hadron is exactly proportional to the square of its energy. One of the rejected models attempts to explain the hadrons as sets of smaller particles held together by forces similar to the elastic force. It is hoped now that string theory or some of its similarities will lead to a more fundamental understanding of quarks and therefore of all matter. String theory to date is considered only a theory, an idea, even with all scientific advances over the last century; it has not yet been possible to verify it experimentally. The expectation is, for the time being, the advances of studies that have been done with particle accelerators.

FINAL CONSIDERATIONS

In order to understand the phenomena that surround us many times it is necessary to construct models, find solutions and validate them. In the study of the oscillatory movement, several activities can be developed with the students in relation to discipline General Physics, even based on interdisciplinary. These activities involve the observation, interaction or even construction of scale models associated to the mass-spring structure, simple or coupled systems and/or pendulums.

Many of the applications of mathematics to experimental sciences and engineering involve differential equations. Most physical problems are mathematically modeled by equations, or systems of equations, that involve partial derivatives of the unknown function. This is due to physical quantities, which are often functions of more than one variable, such as the propagation of a wave that can vary from point to point, in the middle, and depend on time. The rates of change of these quantities are represented by their partial derivatives. It is therefore necessary to present solutions to these equations. In some cases it is possible to solve them by obtaining the so-called analytical solutions and, for example, the Fourier Method is a powerful tool to determine this solution. By considering that students at this level of education do not have knowledge of differential equations, the use of the qualitative study of the solution and the use of numerical methods of approximation of solutions will represent a good tool in solving problems. In a course of Mathematical Physics the student will see more complete formal methods for the solution of the wave equation, but the intention here was to show a way to express the general solution of the wave equation that was obtained by d'Alembert in 1747.

As GREENE (5) explains, string theory is not just a theory of strong force; it is a quantum theory that also includes gravity. It proposes to join the theory of gravitational force and quantum mechanics. The superstring theory says that there is something smaller and more fundamental: inside the quarks, of the smallest subatomic particle, there is a filament of energy that is associated with vibrations like the strings of a violin. And it is the different vibration patterns of these strings that determine the nature of different types of sub particles. This would make it possible to unify the general theory of relativity with quantum mechanics. However, many scientists believe ropes are only mathematical objects.

According to GREENE (5), as polemical as it may seem, string theory is the only candidate that promises us this unification imagined by Einstein and many other scientists of the nineteenth and twentieth centuries, being associated with quantum gravity and superconductivity, in addition to offer many tools related to the most varied mathematical applications. The new idea proposed to understand the protons and neutrons is to present them as a result of vibrations of tiny strings, presenting, even, quantum effects. However, it has been found that the consistency of the new quantum theory of gravity indicates that the space in which we live does not have four dimensions. GREENE (5) states that it was proposed that our universe could have additional dimensions.

Theoretically, it is speculated that space-time must have 10 dimensions. The idea is that the extra dimensions may be so small that they escape all our perceptions and measurements. It has also been discovered that string theory predicts that weak, strong and electromagnetic forces must be associated with a force that is related to its own mediating particles. Clustering all particles and all forces into a single theory is an achievement that no other theory or model can boast. Although interesting, only a few scientists have followed the path of string theory for many decades, however, this study is currently one of the most interesting fields in modern physics, capable of providing the much-desired theory of everything (GREENE, 2001).

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